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LETTER TO THE EDITOR

Off-critical integrable vertex models and conformal theories in finite geometries

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Abstract. We calculate the order parameters (local height probabilities) in the ordered phase of integrable SU(n) vertex models. We show that they have formally the same expression as the partition functions of the associated critical theory in a finite box with appropriate boundary conditions, once the distance to criticality in the former case is properly identified with the modular parameter in the latter. This points out a relation between off-critical integrable models and conformal theories in a finite geometry.

In the last few years, our understanding of two-dimensional critical phenomena has considerably increased through the systematic application of conformal invariance [1]. The most challenging open problem now is probably the unification of these developments with the theory of integrable systems [2–4]. In particular, for a given (solvable) conformal field theory one knows [5, 6] (in general) a lattice model whose critical point corresponds to this theory, but which is integrable for all temperatures. A natural question to ask, then, is the relation between this generic integrability and the conformal invariance which appears at a special point only. Remarkably, it was observed in [7] that the order parameters (local height probabilities) in the low-temperature phase (regime III) of the integrable models associated with the various discrete series of conformal theories have expressions which merely parallel the Goddart *et al* [8] construction for characters of these theories. This suggests that the off-critical integrable direction is deeply related to the critical point, and should correspond to some simple deformation of the conformal symmetry.

In this letter we consider the case of SU(n) vertex models. These have an integrable curve in the parameter space, Boltzmann weights being parametrised by trigonometric, rational or hyperbolic functions. The first region corresponds to a critical model and terminates at the rational point which is described by an SU(n) Wess–Zumino–Witten (WZW) theory [9], while the second region is ordered. We calculate in the latter case the local height probabilities and show that they have the same expression as the (continuum limit) partition functions at the WZW point in a box with appropriate boundary conditions. In this case, going off-criticality in the integrable direction is thus equivalent to putting the critical theory in a finite geometry. The correspondence between distance to T_c and finite size ρ is non-universal, but satisfies $\rho \propto \xi$ (where ξ is the correlation length) when approaching the critical point.

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We start by considering the fundamental SU(2) vertex model, i.e. the six-vertex model [4]. This is defined by putting arrows on the bonds of the square lattice in such a way that the current is conserved at each node, thus giving rise to six possible vertex configurations (figure 1). Imposing invariance under reversal of all arrows, one is left with three free parameters, the Boltzmann weights α, β, $\gamma \geq 0$. Transfer matrices with the same value of $\Delta = (\alpha^2 + \beta^2 - \gamma^2)/2\alpha\beta$ commute, and in the very anisotropic limit they give rise to the spin-$\frac{1}{2}$ XXZ antiferromagnetic quantum chain with Hamiltonian

$$\mathcal{H}\alpha - \sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \Delta \sigma_j^z \sigma_{j+1}^z). \tag{1}$$

In the following it will be convenient to transform this model into an (unrestricted) solid-on-solid model [10]. This is simply done by introducing height variables $\varphi \in \mathbb{Z}$ on the dual lattice in such a way that neighbouring φ differ by ± 1, depending on the arrow which separates them. The region $\Delta > 1$ is frozen with vertices of one kind (1–4) only, corresponding to a totally stretched surface. The region $|\Delta| \leq 1$ is the rough phase; the model is then critical and renormalises onto a Gaussian model with action

$$\mathcal{A} = \frac{\pi g}{4} \int |\nabla \varphi|^2 \, \mathrm{d}^2 x \tag{2}$$

and hence the central charge is $c = 1$. Setting $\Delta = -\cos \lambda$, $\lambda \in [0, \pi]$, one has (in the scale where topological defects are not renormalised) [11]

$$g = 1 - \lambda/\pi \tag{3}$$

while the Boltzmann weights can be parametrised by trigonometric functions

$$\alpha = \sin \lambda \, (1 - u) \qquad \beta = \sin \lambda \, u \qquad \gamma = \sin \lambda \tag{4}$$

where u is the spectral parameter.

The point $\Delta = -1$ corresponds to the roughening transition [10]. The generic U(1) symmetry is then enhanced into an SU(2) symmetry (a sign of which is the isotropy of (1)) and accordingly the model (2) becomes equivalent to the SU(2) level-1 WZW model [9]. The region $\Delta < -1$ is the flat phase (here λ becomes purely imaginary $\lambda = \mathrm{i}\tilde{\lambda}$, and the weights are parametrised by hyperbolic functions). A ground-state configuration has then the same height, say b, on the centre site and those separated from it by even steps, while all other heights assume another fixed value, say c ($c = b \pm 1$). The local height probability (LHP) $P(a/b, c)$, i.e. the probability of finding the central site in state a while the boundary heights are fixed to those of the ground state specified by b and c, has been calculated [12]

$$P(a/b, c) = \frac{p^{[(b+c)/2 - a]^2/4}}{\sum_{a'} p^{[(b+c)/2 - a']^2/4}}. \tag{5}$$

In this expression $p = \exp(-4\tilde{\lambda})$, $\Delta = -\cosh \tilde{\lambda}$.

Equation (5) involves a quadratic dependence upon the difference in heights which is reminiscent of the Gaussian model (2), and this correspondence can be made more

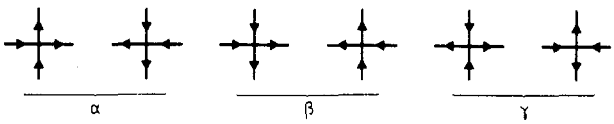

Figure 1. Vertices of the six-vertex model.

precise. Indeed we can consider the partition function of the above solid-on-solid model in the rough phase in a finite geometry consisting of a rectangle $T \times L$ with sides identified in the time direction, while heights on the bottom line $\mathscr{L}_1$ have the same value a, and heights on the two top lines $\mathscr{L}_2$ (resp. $\mathscr{L}_3$) are equal to b (resp. c) (figure 2). (The parity of L must of course be adjusted to the one of $c-a$.) Fixing heights along $\mathscr{L}_1$ is easily realised by keeping there only external vertices which conserve the current. Fixing heights along $\mathscr{L}_2$ and $\mathscr{L}_3$ is realised in the same way by imposing that all the corresponding external vertices be the same. Finally the height difference between top and bottom must be $c-a$.

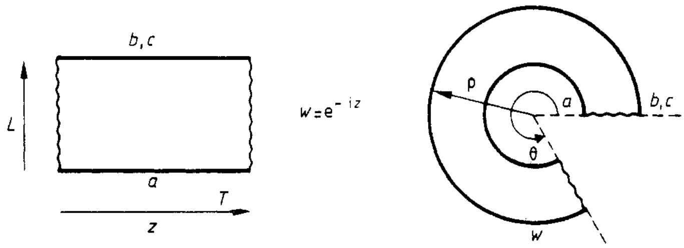

Figure 2. The finite box considered can be mapped onto an annulus in the plane which is reminiscent of the geometry used in the calculation of local height probabilities.

To calculate the partition function of such a system one would like to use the mapping onto (2). The simplest situation arises when heights are fixed along $\mathscr{L}_1$ and $\mathscr{L}_3$ only. In this case one finds in the continuum limit (i.e. when T, $L \to \infty$ keeping T/L fixed)

$$Z(a/c) = \int_{\varphi(x+T,y)=\varphi(x,y),\varphi(x,0)=a,\varphi(x,L)=c} [\mathrm{d}\varphi] \exp(-\mathscr{A}). \tag{6}$$

The functional integral with $a = c$ is calculated by zeta regularisation, and the height difference taken into account by introducing the classical field $\varphi_{\mathrm{cl}} = (c-a)y/L$ to give [13]

$$Z(a/c) = \frac{q^{g(c-a)^2/4}}{\eta(q)} \tag{7}$$

where $q = \exp(-\pi T/L)$ and $\eta(q) = q^{1/24} \Pi_1^\infty (1-q^n)$ is Dedekind's function. Fixing heights on both $\mathscr{L}_2$ and $\mathscr{L}_3$ is more difficult to handle. To derive the corresponding shift on the ground state one can use Bethe ansatz calculations on the XXZ chain (1) with fixed σ_L^z, following [14]. The result is

$$Z(a/b, b+1) = \frac{q^{g[b+1-(1/2g)-a]^2/4}}{\eta(q)} \tag{8a}$$

$$Z(a/b, b-1) = \frac{q^{g[b-1+(1/2g)-a]^2/4}}{\eta(q)}. \tag{8b}$$

These two expressions become identical at the higher symmetry $g = 1$ point, giving then

$$Z(a/b, c) = \frac{q^{[(b+c)/2-a]^2/4}}{\eta(q)}. \tag{9}$$

Remarkably (5) and (9) are completely similar and the LHP in the flat phase can be formally calculated as

$$P(a/b, c) = \frac{Z(a/b, c)}{\Sigma_{a'} Z(a'/b, c)} (q \equiv p). \tag{10}$$

This is a first example where some properties of an integrable model off-criticality and in the thermodynamic limit can be related to those of the same model at criticality but in a finite (continuum) geometry. Instead of the above box, it is perhaps more natural to consider, after conformal mapping on the plane, an annulus with heights fixed to a at the centre and to b, c at the boundary, the outer and inner circles being in the ratio ρ (figure 2). A natural value of T is then 2π, while $L = \ln \rho$. Of course the correspondence (9) needs the precise identification of the distance to criticality with the modular parameter. The model is critical for $\tilde{\lambda} \to 0$, i.e. $p \to 1$; with the correspondence $q = p$, ρ behaves in this limit like

$$\rho \propto \exp \pi^2/2\tilde{\lambda}. \tag{11}$$

It is interesting to notice that (11) is precisely the formula for the correlation length of the model in the same limit, i.e. $\rho \propto \xi$ [4]. This latter relation is expected since from finite scaling an order parameter which scales like $m \propto \xi^a$ for an infinite system close to criticality will behave like $m \propto \rho^a$ at criticality in the geometry of figure 2.

We turn now to the higher SU(2) level-k vertex models. They are obtained [15] by assigning to each bond of the square lattice one of $k+1$ possible states (which can be conveniently represented by arrows carrying a current $j = -k, -k+2, \ldots, k$; see figure 3) with a conservation rule at each node. If $k = 2$, one gets for instance the 19-vertex model of [16], if $k = 3$ a 44-vertex model, ... In the rather large space of parameters, an integrable curve is known, which is obtained starting from the six-vertex ($k = 1$) model by a fusion procedure [15]. Schematically, to obtain the Boltzmann weights one considers a $k \times k$ cell of the level-1 model with well defined inhomogeneities but all vertices having the same value of Δ, one sums over all internal variables, and one projects outer variables onto fully symmetric tensors. In the very anisotropic limit, the transfer matrices define a spin-$\frac{1}{2}k$ XXZ quantum chain [17].

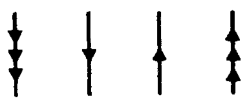

Figure 3. Bond states in the SU(2) level-3 model, carrying respectively a current $j = -3, -1, 1, 3$. They are associated with jumps j of the variable φ, and to rotations $\exp[(2i\pi/3)(j+3)/2]$ of the $\mathbb{Z}_3$ variable.

One can reinterpret the model as an unrestricted solid-on-solid model as well, with height variables $\varphi \in \mathbb{Z}$ on the dual lattice, neighbouring φ differing now by an amount $-k, -k+2, \ldots, k$ which depends on the bond that separates them. The phase diagram is directly deduced from the one of the six-vertex model. The first interesting region is the rough phase where the model is critical and Boltzmann weights are parametrised by trigonometric functions. The associated conformal theory was found in [18]. The heights φ contribute to a bosonic part; on the other hand, one can consider the additional bond-degrees of freedom as associated to a $\mathbb{Z}_k$ model low-temperature expansion, a current j corresponding to a rotation of $\exp[(2i\pi/k)(j+k)/2]$ of the $\mathbb{Z}_k$

variable. In the continuum limit these two aspects decouple (except for boundary condition effects) and one gets a theory which is the product of a free bosonic field times a $\mathbb{Z}_k$ parafermion [19] (in the 19-vertex model case, for instance, one gets a superfree field) with central charge $c=3k/(k+2)$.

The coupling constant g of the bosonic part is known [18]

$$g=1/k-\lambda/\pi \tag{12}$$

where λ parametrises the underlying six-vertex model of the fusion. The roughening transition occurs at $\lambda=0$, $g=1/k$ where the symmetry is enhanced to SU(2) and the model becomes equivalent to the SU(2) level-k WZW model. After this point one enters a flat phase with weights parametrised by hyperbolic functions, and ground states still characterised by a pair of heights b, c. We have calculated the corresponding local height probabilities using results of [5] in which Jimbo *et al* consider restricted models where heights φ belong to a finite set. It is then sufficient to take the appropriate limit of their result, in a way similar to [12], to find

$$P(a/b, c)=\frac{p^{[(b+c)/2-a]^2/4k}C^l_m(p)}{\sum_{a'} p^{[(b+c)/2-a']^2/4k}}. \tag{13}$$

In this formula $p=\exp(-4\tilde{\lambda})$, where $\tilde{\lambda}$ parametrises the underlying six-vertex model of the fusion. $l=(c-b+k)/2$, $m=c-a+k-l \bmod 2k$ and C^l_m is the SU(2) level-k string function [20]. We have assumed without loss of generality that the centre and the boundary lie on different sublattices, which ensure that l and m have the same parity, and thus $C^l_m \neq 0$.

Now we can compare (13) to the partition function of the associated critical model at the roughening point in a box $T\times L$ with heights fixed to a on $\mathscr{L}_1$ and b (resp. c) on $\mathscr{L}_2$ (resp. $\mathscr{L}_3$) as above. Since the conformal theory is a product of a free bosonic field and a $\mathbb{Z}_k$ parafermion, we know [21] that this partition function has to be the product of some $q^h/\eta(q)$ times a string function, or may be a combination of such terms. Remembering (9) and (12) we expect for the bosonic part a contribution

$$Z_B=\frac{q^{[(b+c)/2-a]^2/4k}}{\eta(q)}. \tag{14}$$

In the case $b=c$ which is allowed if k is even this reproduces in particular (7) with the appropriate coupling constant $g=1/k$. The remaining part is more delicate to handle. A simple case is when $b-c=\pm k$. Then, fixing the boundary vertices corresponds simply to having fixed values at the boundary for the parafermionic degrees of freedom. From top to bottom, since a bond with current j corresponds to a rotation $\exp[(2\mathrm{i}\pi/k)(j+k)/2]$, and since $\mathscr{L}_1$ and $\mathscr{L}_3$ have been chosen to lie on different sublattices so an odd number of bonds has been crossed, the total rotation of the $\mathbb{Z}_k$ variable is $\exp[(2\mathrm{i}\pi/k)(c-a+k)/2]$. In this case the partition function can be shown to be [21], generalising [22],

$$Z_{\mathrm{PF}}=C^0_{c-a+k \bmod 2k}=C^0_{a-c+k \bmod 2k}. \tag{15}$$

In writing this last equality the symmetry

$$C^l_m=C^l_{m+2k}=C^l_{-m} \tag{16a}$$

was used. Recall also that

$$C^l_m=C^{k-l}_{k-m}=C^{k-l}_{k+m}. \tag{16b}$$

In the general case Z_{PF} will be a combination of string functions, the indices of which depend on $c-a$ and on the state of the bond fixed at the top of the system, i.e. $(c-b+k)/2$. To avoid any spurious periodicities in the final result, it is reasonable to assume that the dependence of the indices is *linear.* Then using (15) we are necessarily left with $l=\alpha(c-b+k)/2$, $m=c-a+\beta(c-b+k)/2+k$. Moreover if $b=c-k$ (15) must be recovered, which can happen only through (16*b*), with $\alpha=1$, $\beta=-1$. Thus

$$Z_{\mathrm{PF}}=C^{(c-b+k)/2}_{c-a+k-(c-b+k)/2}. \tag{17}$$

As expected (17) is invariant under 'complex conjugation' $(c-b+k)/2\to k-(c-b+k)/2$, $c-a+k\to k-(c-a)$. Collecting (14) and (17) we find

$$Z(a/b,c)=\frac{q^{[(b+c)/2-a]^2/4k}}{\eta(q)}\,C^{(c-b+k)/2}_{c-a+k-(c-b+k)/2}. \tag{18}$$

This formula has the same structure as the local height probability (13), so (10) holds true for SU(2) level-k models as well. On the other hand, we have calculated the singular behaviour of the correlation length when approaching the roughening point using results of [16]. Here also $\xi\propto\exp\pi^2/2\tilde\lambda$ so the correspondence $q\equiv p$ satisfies again $\rho\propto\xi$ in this limit.

Finally, we turn to the case of SU(n) (level-1) vertex models [23]. These are defined by associating to each bond of the square lattice one of the variables $\hat e_i=\sqrt2(\Lambda^i-\Lambda^{i-1})$, where the Λ^i are the fundamental weights of SU(n), and we have set $\Lambda^0=\Lambda^{n+1}=0$ (figure 4). The possible vertices are shown in figure 5, with the integrable weights

$$\begin{aligned}\alpha&=\sin\lambda\ (1-u)\\ \beta&=\sin\lambda\ u\\ \gamma&=\sin\lambda\ \exp[\lambda u\ \mathrm{sgn}(i-j)].\end{aligned} \tag{19}$$

(In the case $n=2$, one recovers (4), due to the fact [4] that there are then as many vertices of the third type with $i>j$ as with $i<j$.) The model can be transformed into

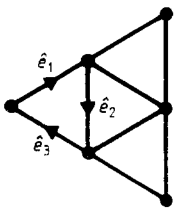

Figure 4. Bond states in the SU(3) model, together with a part of the weight lattice.

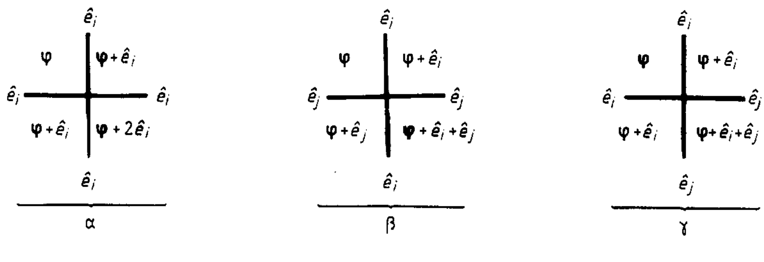

Figure 5. Allowed vertex and associated configurations of the φ variable in the SU(n) model.

an unrestricted solid-on-solid one by introducing on the dual lattice height variables φ which take values in $\mathbb{P}$ equal to $\sqrt{2}$ times the weight lattice of $SU(n)$. If λ in (19) is real, one has a critical rough phase that renormalises onto an $(n-1)$-component free field [24] with action similar to (2) and g still given by (3) [25]. $\lambda=0$ is the roughening point, where the symmetry is enhanced from $U(1)^{n-1}\times\mathbb{Z}_n$ to $SU(n)$, and the model is equivalent to the $SU(n)$ level-1 WZW model. The region $\lambda=\mathrm{i}\tilde{\lambda}$ purely imaginary is a flat phase. The ground-state configurations are then fundamental cells of $\mathbb{P}$ described clockwise, i.e. $\boldsymbol{b}_1, \boldsymbol{b}_2=\boldsymbol{b}_1+\hat{e}_1, \ldots, \boldsymbol{b}_n=\boldsymbol{b}_{n-1}+\hat{e}_n$, and are characterised by any two successive points of the cell $\boldsymbol{b}_i, \boldsymbol{b}_{i+1}$. The corresponding local height probabilities can be obtained by taking the appropriate limit of the results of [6]

$$P[\boldsymbol{a}/\boldsymbol{b}_i, \boldsymbol{b}_{i+1}]=\frac{p^{|\frac{1}{n}\Sigma_{j=i}^{n}(\boldsymbol{b}_j-\boldsymbol{a})|^2/4}}{\Sigma_{\boldsymbol{a}'}\,p^{|\frac{1}{n}\Sigma_{j=i}^{n}(\boldsymbol{b}_j-\boldsymbol{a}')|^2/4}} \tag{20}$$

where $p=\exp(-2n\tilde{\lambda})$. On the other hand we can consider the partition function on the box $T\times L$ in the rough phase. As in (7), if heights were fixed along $\mathscr{L}_1$ and $\mathscr{L}_3$ only, this would be

$$Z(\boldsymbol{a}/\boldsymbol{c})=\frac{q^{g|\boldsymbol{c}-\boldsymbol{a}|^2/4}}{[\eta(q)]^{n-1}}. \tag{21}$$

Fixing heights along $\mathscr{L}_1$ and $\mathscr{L}_2$ induces correction terms in the exponent of q as in (8). We expect all those to give a single expression at the higher symmetry point $g=1$, i.e.

$$Z[\boldsymbol{a}/\boldsymbol{b}_i, \boldsymbol{b}_{i+1}]=\frac{q^{|\frac{1}{n}\Sigma_{j=i}^{n}(\boldsymbol{b}_j-\boldsymbol{a})|^2/4}}{[\eta(q)]^{n-1}}. \tag{22}$$

We see now that relation (10) is also valid. This time, q must be identified with $p=\exp(-2n\tilde{\lambda})$. The correlation length on the other hand was calculated in [24] and diverges like $\xi\propto\exp(\pi^2/n\tilde{\lambda})$, so $\rho\propto\xi$ still holds true.

In the cases we have considered, and as far as order parameters are concerned, going off-criticality in the integrable direction is thus equivalent to putting the critical conformal theory in a finite box. This observation provides a new relation between integrability and conformal invariance. It would be most desirable to extend it to restricted models [26].

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